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Abstract
Associated Legendre functions arise in many problems of mathematical physics. By using the generalized Abel–Plana formula, in this paper we derive a summation formula for the series over the zeros of the associated Legendre function of the first kind with respect to the degree. The summation formula for the series over the zeros of the Bessel function, previously discussed in the literature, is obtained as a limiting case. The Wightman function for a scalar field with a general curvature coupling parameter is considered inside a spherical boundary on the background of a constant negative curvature space. The corresponding mode sum contains the series over the zeros of the associated Legendre function. The application of the summation formula allows us to present the Wightman function in the form of the sum of two integrals. The first one corresponds to the Wightman function for the bulk geometry without boundaries and the second one is induced by the presence of the spherical shell. For points away from the boundary the latter is finite in the coincidence limit. In this way the renormalization of the vacuum expectation value of the field squared is reduced to that for the boundary-free part.

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1. Introduction

In a number of problems in mathematical physics we need to sum over the values of a certain function at integer points, and then subtract the corresponding integral. In particular, in quantum field theory the expectation values for physical observables induced by the presence of boundaries are presented in the form of this difference. The corresponding sum and integral,

taken separately, diverge and some physically motivated procedure for handling the finite result is needed. For a number of boundary geometries one of the most convenient methods for obtaining such renormalized values is based on the use of the Abel–Plana summation formula [1, 2] (for different forms of this formula discussed in the literature see also [3]). Applications of the Abel–Plana formula in physical problems related to the Casimir effect for flat boundary geometries and topologically, non-trivial spaces with corresponding references can be found in [4, 5]. The use of this formula allows us to extract, in a cutoff independent way, the Minkowski vacuum part and to obtain for the renormalized part rapidly convergent integrals useful, in particular, for numerical calculations.

However, the applications of the Abel–Plana formula in its standard form are restricted to the problems where the normal modes are explicitly known. In [6] we have considered a generalization of this formula, which essentially enlarges the application range and allows us to include problems where the eigenmodes are given implicitly as zeros of a given function. Well-known examples of this kind are the boundary-value problems with spherical and cylindrical boundaries. The generalized Abel–Plana formula contains two meromorphic functions and by specifying one of them the Abel–Plana formula is obtained (for other generalizations of the Abel–Plana formula see [5, 7, 8]). Applying the generalized formula to Bessel functions, in [6, 9] summation formulae are obtained for the series over the zeros of various combinations of these functions (for a review with physical applications see also [3, 10, 11]).

The summation formulae derived from the generalized Abel–Plana formula have been applied for the evaluation of the vacuum expectation values of local physical observables in the Casimir effect (for the Casimir effect see [4, 5, 12]) for plane boundaries with Robin or non-local boundary conditions [13], for spherical boundaries in Minkowski and global monopole bulks [14], and for cylindrical boundaries in Minkowski and cosmic string bulks [15]. By making use of the generalized Abel–Plana formula, the vacuum expectation values of the field squared and the energy–momentum tensor in closely related but more complicated geometry of a wedge with cylindrical boundary are investigated in [16] for both scalar and electromagnetic fields. As in the case of the Abel–Plana formula, the use of the generalized formula in these problems allows us to extract the contribution of the unbounded space and to present the boundary-induced parts in terms of exponentially converging integrals. In [17], summation formulae for the series over the zeros of the modified Bessel functions with an imaginary order are derived by using the generalized Abel–Plana formula. This type of series arises in the evaluation of the vacuum expectation values induced by plane boundaries uniformly accelerated through the Fulling–Rindler vacuum. Another class of problems where the application of the generalized Abel–Plana formula provides an efficient way for the evaluation of the vacuum expectation values is considered in [18]. In these papers, braneworld models with two parallel branes on an anti-de Sitter bulk are discussed. The corresponding mode-sums for physical observables bilinear in the field contain the series over the zeroes of cylinder functions which are summarized by using the generalized Abel–Plana formula. The geometry of spherical branes in Rindler-like spacetimes is considered in [19]. In [20] from the generalized Abel–Plana formula a summation formula is derived over the eigenmodes of a dielectric cylinder and this formula is applied for the evaluation of the radiation intensity from a point charge orbiting along a helical trajectory inside the cylinder.

The physical importance of the Bessel functions is related to the fact that they appear as solutions of the field theory equations in various situations. In particular, in spherical and cylindrical coordinates the radial parts of the solutions for the scalar, fermionic and electromagnetic waveequations on the background of the Minkowski spacetime are expressed in terms of these functions. Another important class of special functions is the so-called Legendre associated functions (see, for instance, [21, 22]). These functions can

be considered as generalizations of the Bessel functions: in the limit of large values of the degree when the argument is close to unity they reduce to the Bessel functions. The associated Legendre functions arise naturally in many mathematical and physical applications. In particular, they appear as solutions of physical field equations on the background of constant curvature spaces (see, for instance, [4, 5, 23]) and the above-mentioned limit corresponds to the limit when the curvature radius of the bulk goes to infinity. The eigenfunctions in braneworld models with de Sitter and anti-de Sitter branes are also expressed in terms of the Legendre functions (see [24]). Motivated by this, in the present paper, by making use of the generalized Abel–Plana formula, we obtain a summation formula for the series over the zeros of the associated Legendre function of the first kind with respect to the degree. In particular, this type of series appears in the evaluation of expectation values for physical observables bilinear in the operator of a quantum field on background of constant curvature spaces in the presence of boundaries. As in the case of the other Abel–Plana-type formulae, previously considered in the literature, the formula discussed here presents the sum of the series over the zeros of the associated Legendre function in the form of the sum of two integrals. In boundary-value problems the first one corresponds to the situation when the boundary is absent and the second one presents the part induced by the boundary. For a large class of functions the latter is rapidly convergent and, in particular, is useful for the numerical evaluations of the corresponding physical characteristics.

We have organized the paper as follows. In section 2, by specifying the functions in the generalized Abel–Plana formula we derive a formula for the summation of the series over the zeros of the associated Legendre function with respect to the degree. In section 3, special cases of this summation formula are considered. First, as a partial check we show that, as a special case, the standard Abel–Plana formula is obtained. Then we show that from the summation formula discussed in section 2, as a limiting case the formula is obtained for the summation of the series over the zeros of the Bessel function, previously derived in [6]. A physical application is given in section 4, where the positive frequency Wightman function for a scalar field is evaluated inside a spherical boundary on the background of a negative constant curvature space. It is assumed that the field obeys the Dirichlet boundary condition on the spherical shell. The use of the summation formula from section 2 allows us to extract from the vacuum expectation value the part corresponding to the geometry without boundaries and to present the part induced by the spherical shell in terms of an integral, which is rapidly convergent in the coincidence limit for points away from the boundary. The main results of the paper are summarized in section 5. In appendix A we show that the zeros of the associated Legendre function of the first kind with respect to the degree are simple and real, and the asymptotic form for large zeros is discussed. In appendix B, asymptotic formulae for the associated Legendre functions are considered for large values of the degree. These formulae are used in section 2 to obtain the constraints imposed on the function appearing in the summation formula.

2. Summation formula

In this section we derive a summation formula for the series over zeros of the associated Legendre function of the first kind, $P^{\mu}_{iz-1/2}(u)$, with respect to the degree, assuming that $u > 1$ and $\mu \leqslant 0$ (in this paper the definition of the associated Legendre functions follows that given in [22]). For given values u and μ this function has an infinity of real zeros. We will denote the positive zeros arranged in ascending order of magnitude as z_k:

$$P^{\mu}_{iz_k-1/2}(u) = 0, \qquad k = 1, 2, \ldots. \tag{1}$$

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These zeros are functions of the parameters u and μ: $z_k = z_k(u, \mu)$. Note that one has $P^{\mu}_{\mathrm{i}z-1/2}(u) = P^{\mu}_{-\mathrm{i}z-1/2}(u)$ and, hence, $-z_k$ are zeros of the function $P^{\mu}_{\mathrm{i}z-1/2}(u)$ as well. In appendix A we show that the zeros z_k are simple and under the conditions specified above the function $P^{\mu}_{\mathrm{i}z-1/2}(u)$ has no zeros which are not real.

A summation formula for the series over z_k can be obtained by making use of the generalized Abel–Plana formula [6] (see also [3])

$$\lim_{b \to \infty} \left\{ \text{p.v.} \int_a^b \mathrm{d}x \, f(x) - R[f(z), g(z)] \right\} = \frac{1}{2} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \mathrm{d}z [g(z) + \sigma(z) f(z)], \tag{2}$$

where $\sigma(z) \equiv \mathrm{sgn}(\mathrm{Im}\, z)$, the functions $f(z)$ and $g(z)$ are meromorphic for $a \leqslant x \leqslant b$ in the complex plane $z = x + \mathrm{i}y$ and p.v. stands for the principal value of the integral. In formula (2) we have defined

$$R[f(z), g(z)] = \pi \mathrm{i} \left[\sum_k \mathrm{Res}_{z=z_{g,k}} g(z) + \sum_{k, \mathrm{Im}\, z_{f,k} \neq 0} \sigma(z_{f,k}) \mathrm{Res}_{z=z_{f,k}} f(z) \right], \tag{3}$$

with $z_{f,k}$ and $z_{g,k}$ being the positions of the poles of the functions $f(z)$ and $g(z)$ in the strip $a < x < b$.

We choose the functions $f(z)$ and $g(z)$ in formula (2) in the form

$$
\begin{aligned}
f(z) &= \sinh(\pi z) h(z), \\
g(z) &= \frac{\mathrm{e}^{-\mathrm{i}\mu\pi} h(z)}{\pi \mathrm{i} P^{\mu}_{\mathrm{i}z-1/2}(u)} \left\{ \cos[\pi(\mu + \mathrm{i}z)] Q^{\mu}_{\mathrm{i}z-1/2}(u) + \cos[\pi(\mu - \mathrm{i}z)] Q^{\mu}_{-\mathrm{i}z-1/2}(u) \right\},
\end{aligned}
\tag{4}
$$

where $Q^{\mu}_{\mathrm{i}z-1/2}(u)$ is the associated Legendre function of the second kind and the function $h(z)$ is meromorphic for $a \leqslant \mathrm{Re}\, z \leqslant b$. By using relation (B.3) between the associated Legendre functions given in appendix B, for the combination appearing on the left-hand side of formula (2), one finds

$$g(z) \pm f(z) = \frac{2 \, \mathrm{e}^{-\mathrm{i}\mu\pi} h(z)}{\pi \mathrm{i} P^{\mu}_{\mathrm{i}z-1/2}(u)} \cos[\pi(\mu \mp \mathrm{i}z)] Q^{\mu}_{\mp \mathrm{i}z-1/2}(u). \tag{5}$$

With the functions (4) the expression for $R[f(z), g(z)]$ takes the form

$$R[f(z), g(z)] = 2 \sum_k \left. \frac{\mathrm{e}^{-\mathrm{i}\mu\pi} Q^{\mu}_{\mathrm{i}z-1/2}(u)}{\partial_z P^{\mu}_{\mathrm{i}z-1/2}(u)} \cos[\pi(\mu + \mathrm{i}z)] h(z) \right|_{z=z_k} + 2 \, \mathrm{e}^{-\mathrm{i}\mu\pi} r[h(z)], \tag{6}$$

with the notation

$$
\begin{aligned}
r[h(z)] = {}& \sum_{k, \mathrm{Im}\, z_{h,k} \neq 0} \mathrm{Res}_{z=z_{h,k}} \left\{ \frac{Q^{\mu}_{-\sigma(z)\mathrm{i}z-1/2}(u)}{P^{\mu}_{\mathrm{i}z-1/2}(u)} \cos[\pi(\mu - \sigma(z)\mathrm{i}z)] h(z) \right\} \\
& + \frac{1}{2} \sum_{k, \mathrm{Im}\, z_{h,k} = 0} \mathrm{Res}_{z=z_{h,k}} \left\{ \frac{h(z)}{P^{\mu}_{\mathrm{i}z-1/2}(u)} \sum_{l=\pm} \cos[\pi(\mu + l\mathrm{i}z)] Q^{\mu}_{l\mathrm{i}z-1/2}(u) \right\}.
\end{aligned}
\tag{7}
$$

In formula (7), $z_{h,k}$ are the positions of the poles for the function $h(z)$.

In terms of the function $h(z)$ the conditions for the generalized Abel–Plana formula (2) to be valid take the form

$$
\begin{aligned}
\lim_{w \to \infty} \int_{a \pm \mathrm{i}w}^{b \pm \mathrm{i}w} \mathrm{d}z \, \frac{Q^{\mu}_{\mp \mathrm{i}z-1/2}(u)}{P^{\mu}_{\mathrm{i}z-1/2}(u)} \cos[\pi(\mu \mp \mathrm{i}z)] h(z) &= 0, \\
\lim_{b \to \infty} \int_b^{b \pm \mathrm{i}\infty} \mathrm{d}z \, \frac{Q^{\mu}_{\mp \mathrm{i}z-1/2}(u)}{P^{\mu}_{\mathrm{i}z-1/2}(u)} \cos[\pi(\mu \mp \mathrm{i}z)] h(z) &= 0.
\end{aligned}
\tag{8}
$$

By using the asymptotic formulae for the associated Legendre functions given in appendix B, it can be seen that these conditions are satisfied if the function $h(z)$ is restricted to the constraint

$$|h(z)| < \varepsilon(x)\, e^{c\eta y}, \qquad z = x + iy, \qquad |z| \to \infty, \tag{9}$$

uniformly in any finite interval of x, where $c < 2$, $\varepsilon(x)\, e^{\pi x} \to 0$ for $x \to +\infty$, and η is defined by the relation

$$u = \cosh \eta. \tag{10}$$

Substituting the functions (4) into formula (2) and by taking into account relations (5) and (6), we obtain that for a function $h(z)$ meromorphic in the half-plane $\mathrm{Re}\, z \geqslant a$ and satisfying condition (9), the following formula takes place:

$$\lim_{b\to\infty} \left\{ \sum_{k=m}^{n} T_\mu(z_k, u) h(z_k) - \frac{e^{i\mu\pi}}{2} \mathrm{p.v.} \int_a^b dx \, \sinh(\pi x) h(x) + r[h(z)] \right\}$$
$$= \frac{i}{2\pi} \int_{a-i\infty}^{a+i\infty} dz \, \frac{Q^\mu_{-\sigma(z)iz-1/2}(u)}{P^\mu_{iz-1/2}(u)} \cos[\pi(\mu - \sigma(z)iz)] h(z), \tag{11}$$

where and in what follows the notation

$$T_\mu(z, u) = \frac{Q^\mu_{iz-1/2}(u)}{\partial_z P^\mu_{iz-1/2}(u)} \cos[\pi(\mu + iz)] \tag{12}$$

is used. On the left-hand side of formula (11), $z_{m-1} < a < z_m$, $z_n < b < z_{n+1}$ and in the definition of $r[h(z)]$ the summation goes over the poles $z_{h,k}$ in the strip $a < \mathrm{Re}\, z < b$.

Note that from the Wronskian relation for the associated Legendre functions one has

$$Q^\mu_{iz-1/2}(u) = \frac{e^{i\mu\pi}\Gamma(iz + \mu + 1/2)}{(u^2 - 1)\Gamma(iz - \mu + 1/2)\partial_u P^\mu_{iz-1/2}(u)}, \qquad z = z_k. \tag{13}$$

Now, by taking into account the formula

$$\frac{\Gamma(iz + \mu + 1/2)}{\Gamma(iz - \mu + 1/2)} = \pi \frac{|\Gamma(iz - \mu + 1/2)|^{-2}}{\cos[\pi(\mu + iz)]}, \tag{14}$$

for the gamma function, the factor $T_\mu(z_k, u)$ in (11) can also be written in the form

$$T_\mu(z_k, u) = \frac{\pi e^{i\mu\pi}|\Gamma(iz - \mu + 1/2)|^{-2}}{(u^2 - 1)\partial_u P^\mu_{iz-1/2}(u)\partial_z P^\mu_{iz-1/2}(u)}\bigg|_{z=z_k}.$$

Taking the limit $a \to 0$, from (11) one obtains that for a function $h(z)$ meromorphic in the half-plane $\mathrm{Re}\, z \geqslant 0$ and satisfying the condition (9), the following formula takes place:

$$\sum_{k=1}^{\infty} T_\mu(z_k, u) h(z_k) = \frac{e^{i\mu\pi}}{2} \mathrm{p.v.} \int_0^\infty dx \, \sinh(\pi x) h(x) - r[h(z)]$$
$$- \frac{1}{2\pi} \int_0^\infty dx \, \frac{Q^\mu_{x-1/2}(u)}{P^\mu_{x-1/2}(u)} \cos[\pi(\mu + x)][h(x\, e^{\pi i/2}) + h(x\, e^{-\pi i/2})]. \tag{15}$$

If the function $h(z)$ has poles on the positive real axis, it is assumed that the first integral on the right-hand side converges in the sense of the principal value. From the derivation of (15) it follows that this formula may be extended to the case of some functions $h(z)$ having branch points on the imaginary axis, for example, having the form $h(z) = h_1(z)/(z^2 + c^2)^{1/2}$, where $h_1(z)$ is a meromorphic function. This type of function appears in the physical example discussed in section 4. Special cases of formula (15) with examples are considered in the following section.

Formula (15) can be generalized for a class of functions $h(z)$ having purely imaginary poles at the points $z = \pm iy_k$, $y_k > 0$, $k = 1, 2, \ldots$, and at the origin $z = y_0 = 0$. Let function $h(z)$ satisfy the condition

$$h(z) = -h(ze^{-\pi i}) + o((z - \sigma_k)^{-1}), \qquad z \to \sigma_k, \qquad \sigma_k = 0, iy_k. \qquad (16)$$

Now, in the limit $a \to 0$ the right-hand side of (11) can be presented in the form

$$\frac{i}{2\pi} \sum_{\alpha=+,-} \left(\int_{\gamma_\rho^\alpha} dz + \sum_{\sigma_k = \alpha i y_k} \int_{C_\rho(\sigma_k)} dz \right) \frac{Q^\mu_{-\alpha iz - 1/2}(u)}{P^\mu_{iz-1/2}(u)} \cos[\pi(\mu - \alpha iz)]h(z), \qquad (17)$$

plus the sum of the integrals along the straight segments $(\pm i(y_{k-1} + \rho), \pm i(y_k - \rho))$ of the imaginary axis between the poles. In (17), $C_\rho(\sigma_k)$ denotes the right half of the circle with radius ρ and with the center at the point σ_k, described in the positive direction. Similarly, γ_ρ^+ and γ_ρ^- are upper and lower halves of the semicircle in the right half-plane with radius ρ and with the center at the point $z = 0$, described in the positive direction with respect to this point. In the limit $\rho \to 0$ the sum of the integrals along the straight segments of the imaginary axis gives the principal value of the last integral on the right-hand side of (15). Further, in the terms of (17) with $\alpha = -$ we introduce a new integration variable $z' = ze^{\pi i}$. By using the relation (16), the expression (17) is presented in the form

$$- \sum_{\sigma_k = 0, iy_k} (1 - \delta_{0\sigma_k}/2) \operatorname*{Res}_{z=\sigma_k} \left\{ \frac{Q^\mu_{-iz-1/2}(u)}{P^\mu_{iz-1/2}(u)} \cos[\pi(\mu - iz)]h(z) \right\} \qquad (18)$$

plus the part which vanishes in the limit $\rho \to 0$. As a result, formula (15) is extended for functions having purely imaginary poles and satisfying condition (16). For this, on the right-hand side of (15) we have to add the sum of residues (18) at these poles and take the principal value of the second integral on the right-hand side. The latter exists due to condition (16). Note that for functions having the form $h(z) = F(z)P^\mu_{iz-1/2}(u)$ the left-hand side of (15) is zero and from this formula we obtain a formula relating the integrals involving the Legendre associated functions.

3. Special cases

Here we will consider special cases of the summation formula (15). First let us consider the case $\mu = -1/2$. The corresponding associated Legendre functions have the form

$$P^{-1/2}_{z-1/2}(\cosh\eta) = \sqrt{\frac{2}{\pi}} \frac{\sinh(z\eta)}{z\sqrt{\sinh\eta}}, \quad Q^{-1/2}_{z-1/2}(\cosh\eta) = -i\sqrt{\frac{\pi}{2}} \frac{e^{-z\eta}}{z\sqrt{\sinh\eta}}. \qquad (19)$$

In this case one has $z_k = \pi k/\eta$. Introducing a new function $F(x)$ in accordance with the relation $F(\eta x/\pi) = \sinh(\pi x)h(x)$, and assuming that this function is analytic in the right half-plane, from formula (15) we find the Abel–Plana formula in the standard form:

$$\sum_{k=1}^{\infty} F(k) = -\frac{1}{2}F(0) + \int_0^\infty dx\, F(x) + i\int_0^\infty dx\, \frac{F(ix) - F(-ix)}{e^{2\pi x} - 1}. \qquad (20)$$

Note that the first term on the right-hand side of this formula comes from the residue term with $\sigma_k = 0$ in (18).

In the case $\mu = 1/2$ for the corresponding associated Legendre functions we have the expressions

$$P^{1/2}_{z-1/2}(\cosh\eta) = \sqrt{\frac{2}{\pi}} \frac{\cosh(z\eta)}{\sqrt{\sinh\eta}}, \qquad Q^{1/2}_{z-1/2}(\cosh\eta) = i\sqrt{\frac{\pi}{2}} \frac{e^{-z\eta}}{\sqrt{\sinh\eta}}. \qquad (21)$$

The zeros z_k now have the form $z_k = \pi(k + 1/2)/\eta$ and for functions $F(z)$ analytic in the right half-plane from formula (15) we obtain the Abel–Plana formula in the form useful for fermionic field calculations (see, for instance, [4, 5]):

$$\sum_{k=1}^{\infty} F(k + 1/2) = \int_0^{\infty} \mathrm{d}x\, F(x) - \mathrm{i} \int_0^{\infty} \mathrm{d}x\, \frac{F(\mathrm{i}x) - F(-\mathrm{i}x)}{\mathrm{e}^{2\pi x} + 1}. \tag{22}$$

As a next special case, let us consider the formula for the summation over the zeros of the function $P_{\mathrm{i}sz-1/2}^{-\mu}(\cosh(\eta/s))$ in the limit when $s \to \infty$. By taking into account the relation (see appendix B)

$$\lim_{\nu \to +\infty} \nu^{\mu} P_{\mathrm{i}\nu-1/2}^{-\mu}(\cosh(\eta/\nu)) = J_{\mu}(\eta), \tag{23}$$

with $J_{\mu}(\eta)$ being the Bessel function of the first kind, in this limit from (15) we obtain the summation formula for the series over zeros $\eta = j_{\mu,k}, k = 1, 2, \ldots$, of the Bessel function. In order to take this limit we also will need the formulae (B.10) from appendix B and the formulae [25]

$$\lim_{\nu \to \infty} \nu^{\mu} P_{\nu}^{-\mu}[\cosh(x/\nu)] = I_{\mu}(x), \qquad \lim_{\nu \to \infty} \nu^{\mu} Q_{\nu}^{-\mu}[\cosh(x/\nu)] = \mathrm{e}^{-\mathrm{i}\mu\pi} K_{\mu}(x), \tag{24}$$

with $I_{\mu}(x), K_{\mu}(x)$ being the modified Bessel functions. First we rewrite formula (15) making the replacements $z \to sz, x \to sx, \mu \to -\mu$, on both sides of this formula including the terms in $r[h(z)]$, and we take $u = \cosh(\eta/s)$. In order to take the limit $s \to \infty$ for the second integral on the right-hand side of the resulting formula, we note that, as it follows from the derivation of (15), the integrand of this integral (with the replacements described above) should be understood as the limit

$$\cos[\pi(sx - \mu)] \sum_{l=+,-} h(sx\, \mathrm{e}^{l\pi \mathrm{i}/2}) = \lim_{\epsilon \to +0} \sum_{l=+,-} \cos[\pi(sx - \mu - l\mathrm{i}s\epsilon)]h(sx\, \mathrm{e}^{l\pi \mathrm{i}/2}). \tag{25}$$

Taking the limit $s \to \infty$ with the help of formulae (23), (24), (B.10), we find the following summation formula over the zeros of the Bessel function,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{2f(j_{\mu,k})}{j_{\mu,k} J_{\mu}'^2(j_{\mu,k})} &= \mathrm{p.v.} \int_0^{\infty} \mathrm{d}x f(x) - r_1[f(z)] \\
&\quad - \frac{1}{\pi} \int_0^{\infty} \mathrm{d}x\, \frac{K_{\mu}(x)}{I_{\mu}(x)} \left[\mathrm{e}^{\mathrm{i}\pi\mu} f(x\, \mathrm{e}^{\pi \mathrm{i}/2}) + \mathrm{e}^{-\mathrm{i}\pi\mu} f(x\, \mathrm{e}^{-\pi \mathrm{i}/2}) \right],
\end{aligned} \tag{26}$$

where $f(z) = \lim_{s \to \infty} \mathrm{e}^{sz/\eta} h(sz/\eta)$, and

$$\begin{aligned}
r_J[f(z)] &= \pi \mathrm{i} \sum_k \operatorname*{Res}_{\operatorname{Im} z_{h,k} > 0} \left[\frac{H_{\mu}^{(1)}(z)}{J_{\mu}(z)} f(z) \right] - \pi \mathrm{i} \sum_k \operatorname*{Res}_{\operatorname{Im} z_{h,k} < 0} \left[\frac{H_{\mu}^{(2)}(z)}{J_{\mu}(z)} f(z) \right] \\
&\quad - \pi \sum_k \operatorname*{Res}_{\operatorname{Im} z_{h,k} = 0} \left[\frac{Y_{\mu}(z)}{J_{\mu}(z)} f(z) \right] - \frac{\pi}{2} \operatorname*{Res}_{z=0} \left[\frac{Y_{\mu}(z)}{J_{\mu}(z)} f(z) \right].
\end{aligned} \tag{27}$$

This formula is a special case of the result derived in [6, 9] (see also [3]).

Now let us consider two important special cases of (15) corresponding to $\mu = -l, h(z) = H(z)/\cosh(\pi z)$ and $\mu = -l - 1/2, h(z) = H(z)/\sinh(\pi z)$ with $l = 0, 1, 2, \ldots$. The associated Legendre functions with these values of the order appear as radial solutions of the equations for various fields on the background of constant curvature spaces in cylindrical and spherical coordinates. Let the function $H(z)$ be meromorphic in the half-plane $\operatorname{Re} z \geqslant 0$ and satisfy the condition

$$|H(z)| < \varepsilon_H(x)\, \mathrm{e}^{c\eta y}, \qquad z = x + \mathrm{i}y, \qquad |z| \to \infty, \tag{28}$$

uniformly in any finite interval of $x > 0$, where $c < 2$, $\varepsilon_H(x) \to 0$ for $x \to +\infty$. Then from the results of section 2 it follows that the formula

$$\sum_{k=1}^{\infty} \frac{(-1)^\delta Q_{\mathrm{i}z_k-1/2}^{-l-\delta/2}(u) H(z_k)}{\partial_z P_{\mathrm{i}z-1/2}^{-l-\delta/2}(u)|_{z=z_k}} = \frac{1}{2} \mathrm{p.v.} \int_0^\infty \mathrm{d}x \tanh^{1-\delta}(\pi x) H(x) - r_\delta[H(z)]$$
$$- \frac{1}{2\pi} \int_0^\infty \mathrm{d}x \frac{Q_{x-1/2}^{-l-\delta/2}(u)}{P_{x-1/2}^{-l-\delta/2}(u)} [H(x\,\mathrm{e}^{\pi\mathrm{i}/2}) + (-1)^\delta H(x\,\mathrm{e}^{-\pi\mathrm{i}/2})], \tag{29}$$

takes place, where $\delta = 0, 1$. In this formula we have introduced the notation

$$r_\delta[H(z)] = \sum_{k,\mathrm{Im}\,z_{h,k} \neq 0} \mathrm{Res}_{z=z_{h,k}} \left[\sigma^\delta(z) \frac{Q_{-\sigma(z)\mathrm{i}z-1/2}^{-l-\delta/2}(u)}{P_{\mathrm{i}z-1/2}^{-l-\delta/2}(u)} H(z) \right]$$
$$+ \sum_{k,\mathrm{Im}\,z_{h,k}=0} \mathrm{Res}_{z=z_{h,k}} \left[\frac{Q_{-\mathrm{i}z-1/2}^{-l-\delta/2}(u) + (-1)^\delta Q_{\mathrm{i}z-1/2}^{-l-\delta/2}(u)}{2 P_{\mathrm{i}z-1/2}^{-l-\delta/2}(u)} H(z) \right]. \tag{30}$$

Adding to the right-hand side of formula (29) the term

$$-(-1)^\delta \sum_{\sigma_k=0,\mathrm{i}y_k} (1 - \delta_{0\sigma_k}/2) \mathrm{Res}_{z=\sigma_k} \left[\frac{Q_{-\mathrm{i}z-1/2}^{-l-\delta/2}(u)}{P_{\mathrm{i}z-1/2}^{-l-\delta/2}(u)} H(z) \right], \tag{31}$$

with $H(z)$ obeying the condition $H(z) = -(-1)^\delta H(z\mathrm{e}^{-\pi\mathrm{i}}) + o((z - \sigma_k)^{-1})$ for $z \to \sigma_k$, we obtain the extension of this formula to the case when the function $H(z)$ has poles at the points $0, \pm y_k$.

From (29), as an example when the series is summarized in closed form one has

$$\sum_{k=1}^{\infty} \frac{Q_{\mathrm{i}z-1/2}^{-l-1/2}(u)}{\partial_z P_{\mathrm{i}z-1/2}^{-l-1/2}(u)} \frac{z^{2n} \cos(\alpha z)}{(z^2 + c^2)^{m+1}} \Bigg|_{z=z_k} = \frac{(-1)^{m+n}}{m!} \left\{ -\frac{\pi}{2^{m+2}} \left(\frac{\partial}{c\partial c} \right)^m (c^{2n-1}\mathrm{e}^{-\alpha c}) \right.$$
$$\left. - \mathrm{i} \frac{\partial^m}{\partial x^m} \left[\frac{Q_{x-1/2}^{-l-1/2}(u)}{P_{x-1/2}^{-l-1/2}(u)} \frac{x^{2n} \cosh(\alpha x)}{(x+c)^{m+1}} \right]_{x=c} \right\}, \tag{32}$$

where $\alpha < 2\eta$, $c > 0$, with $m \geqslant 0$ and $0 \leqslant n \leqslant m$ being integers. The last term on the right-hand side of this formula comes from the residue at the pole $\sigma_k = \mathrm{i}c$. As a next example, we take in formula (29) with $\delta = 1$ the function

$$H(z) = z^{2n-\nu} J_\nu(az) \frac{J_\alpha(b\sqrt{z^2 + c^2})}{(z^2 + c^2)^{\alpha/2}}, \tag{33}$$

where a, b and c are positive constants and n is a non-negative integer. This function is analytic in the right half-plane and satisfies the condition (28) if $a + b < 2\eta$, $2n < \alpha + \nu$. By taking into account that (33) is an even function of z, from (29) we find

$$\sum_{k=1}^{\infty} J_\nu(az_k) \frac{J_\alpha\big(b\sqrt{z_k^2 + c^2}\big)}{(z_k^2 + c^2)^{\alpha/2}} \frac{z_k^{\nu+2n} Q_{\mathrm{i}z_k-1/2}^{-l-1/2}(u)}{\partial_z P_{\mathrm{i}z-1/2}^{-l-1/2}(u)|_{z=z_k}} = -\frac{1}{2} \int_0^\infty \mathrm{d}x\, x^{2n-\nu} J_\nu(ax) \frac{J_\alpha(b\sqrt{x^2 + c^2})}{(x^2 + c^2)^{\alpha/2}}. \tag{34}$$

4. The Wightman function inside a spherical boundary in a constant curvature space

In this section we consider a physical application of the summation formula derived in section 2. Namely, we will evaluate the positive frequency Wightman function for a scalar

field and the vacuum expectation value of the field squared inside a spherical shell in a constant negative curvature space assuming that the field obeys the Dirichlet boundary condition on the shell (for quantum effects on background of constant curvature spaces see, for instance, [4, 5, 23] and references therein).

Consider a quantum scalar field $\varphi(x)$ with the curvature coupling parameter ξ on background of the space with the constant negative curvature described by the line element

$$ds^2 = dt^2 - a^2[dr^2 + \sinh^2 r(d\theta^2 + \sin^2\theta d\phi^2)], \tag{35}$$

where a is a constant which is related to the non-zero components of the Ricci tensor and the Ricci scalar by the relations

$$R_1^1 = R_2^2 = R_3^3 = -\frac{2}{a^2}, \qquad R = -\frac{6}{a^2}. \tag{36}$$

The field equation has the form

$$(\nabla_l\nabla^l + M^2 + \xi R)\varphi(x) = 0, \tag{37}$$

where M is the mass of the field quanta.

We are interested in quantum effects induced by the presence of a spherical shell with radius $r = r_0$, on which the field obeys the Dirichlet boundary condition: $\varphi(x)|_{r=r_0} = 0$. This boundary condition modifies the spectrum of the zero-point fluctuations compared with the case of free space and changes the physical properties of the vacuum. Among the most important characteristics of the vacuum are the expectation values of quantities bilinear in the field operator such as the field squared and the energy–momentum tensor. These expectation values are obtained from two-point functions in the coincidence limit. As a two-point function here we will consider the positive frequency Wightman function $W(x, x') = \langle 0|\varphi(x)\varphi(x')|0\rangle$, where $|0\rangle$ is the amplitude for the vacuum state. This function also determines the response of Unruh–De Witt-type particle detectors [23]. Expanding the field operator over the complete set $\{\varphi_\alpha(x), \varphi_\alpha^*(x)\}$ of classical solutions to the field equation satisfying the boundary condition, the Wightman function is presented as the mode-sum

$$W(x, x') = \sum_\alpha \varphi_\alpha(x)\varphi_\alpha^*(x'), \tag{38}$$

where α is a set of quantum numbers specifying the solution.

By the symmetry of the problem under consideration, the eigenfunctions for the scalar field can be presented in the form

$$\varphi_\alpha(x) = Z(r)Y_{lm}(\theta, \phi)\, e^{-i\omega t}, \tag{39}$$

where $Y_{lm}(\theta, \phi)$ are standard spherical harmonics, $l = 0, 1, 2, \ldots, -l \leqslant m \leqslant l$. From the field equation (37) we obtain the equation for the radial function $Z(r)$:

$$\frac{1}{\sinh^2 r}\frac{d}{dr}\left(\sinh^2 r\frac{dZ}{dr}\right) + \left[(\omega^2 - m_{\text{eff}}^2)a^2 - \frac{l(l+1)}{\sinh^2 r}\right]Z = 0, \tag{40}$$

where we have introduced the effective mass defined by

$$m_{\text{eff}}^2 = M^2 - 6\xi/a^2. \tag{41}$$

In the region inside the spherical shell the solution of equation (40), finite at $r = 0$, is expressed in terms of the associated Legendre function of the first kind and the eigenfunctions have the form

$$\varphi_\alpha(x) = C_\alpha \frac{P_{iz-1/2}^{-l-1/2}(\cosh r)}{\sqrt{\sinh r}}Y_{lm}(\theta, \phi)\, e^{-i\omega t}, \tag{42}$$

with the notation

$$z^2 = \left(\omega^2 - m_{\text{eff}}^2\right)a^2 - 1. \tag{43}$$

From the boundary condition on the spherical shell we find that the eigenvalues for z are solutions of the equation

$$P_{iz-1/2}^{-l-1/2}(\cosh r_0) = 0, \tag{44}$$

and, hence, $z = z_k, k = 1, 2, \ldots$, in the notation of section 2. The corresponding eigenfrequencies are found to be

$$\omega_k^2 = \omega^2(z_k) = \left(z_k^2 + 1 - 6\xi\right)\big/a^2 + M^2. \tag{45}$$

Hence, the set α of the quantum numbers is specified to $\alpha = (l, m, k)$.

The coefficient C_α in (42) is determined from the orthonormalization condition

$$\int \mathrm{d}^3x \sqrt{|g|}\varphi_\alpha(x)\varphi_{\alpha'}^*(x) = \frac{\delta_{\alpha\alpha'}}{2\omega}, \tag{46}$$

where the integration goes over the region inside the spherical shell. Substituting the eigenfunctions (42) into (46), by taking into account the integration formula (A.3) and the boundary condition, one finds

$$C_\alpha^{-2} = a^3\frac{\omega(z)}{z}\left(u_0^2 - 1\right)\partial_z P_{iz-1/2}^{-l-1/2}(u_0)\partial_u P_{iz-1/2}^{-l-1/2}(u)|_{z=z_k,u=u_0}, \tag{47}$$

where and in the discussion below we use the notations

$$u = \cosh r, \qquad u_0 = \cosh r_0. \tag{48}$$

By using the Wronskian relation (13), the formula for the normalization coefficient is written as

$$C_\alpha^2 = \frac{z_k\Gamma(iz_k + l + 1)Q_{iz_k-1/2}^{-l-1/2}(u_0)\,\mathrm{e}^{i(l+1/2)\pi}}{a^3\omega(z_k)\Gamma(iz_k - l)\partial_z P_{iz-1/2}^{-l-1/2}(u_0)|_{z=z_k}}. \tag{49}$$

Note that the ratio of the gamma functions in this formula can also be presented in the form

$$\frac{\Gamma(iz_k + l + 1)}{\Gamma(iz_k - l)} = |\Gamma(iz_k + l + 1)|^2\frac{\cos[\pi(iz - l - 1/2)]}{\pi}. \tag{50}$$

Substituting the eigenfunctions into the mode-sum formula (38) and using the addition theorem for the spherical harmonics, for the Wightman function one finds

$$W(x, x') = \frac{1}{4\pi^2a^3}\sum_{l=0}^{\infty}\frac{(2l+1)P_l(\cos\gamma)}{\sqrt{\sinh r \sinh r'}}\,\mathrm{e}^{i(l+1/2)\pi}\sum_{k=1}^{\infty}z_k|\Gamma(iz_k + l + 1)|^2$$

$$\times T_{-l-1/2}(z_k, u_0)P_{iz_k-1/2}^{-l-1/2}(\cosh r)P_{iz_k-1/2}^{-l-1/2}(\cosh r')\frac{\mathrm{e}^{-i\omega(z_k)\Delta t}}{\omega(z_k)}, \tag{51}$$

where $\Delta t = t - t'$ and $T_\mu(z, u)$ is defined by relation (12). In (51), $P_l(\cos\gamma)$ is the Legendre polynomial and

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'). \tag{52}$$

As the expressions for the zeros z_k are not explicitly known, formula (51) for the Wightman function is not convenient. In addition, the terms in the sum are highly oscillatory for large values of quantum numbers.

For further evaluation of the Wightman function we apply to the series over k the summation formula (15) taking in this formula

$$h(z) = z|\Gamma(iz + l + 1)|^2 P_{iz-1/2}^{-l-1/2}(\cosh r)P_{iz-1/2}^{-l-1/2}(\cosh r')\frac{\mathrm{e}^{-i\omega(z)\Delta t}}{\omega(z)}. \tag{53}$$

The corresponding conditions are met if $r + r' + \Delta t/a < 2r_0$. In particular, this is the case in the coincidence limit $t = t'$ for the region under consideration. For the function (53) the part of the integral on the right-hand side of formula (15) over the region $(0, x_M)$ vanishes and for the Wightman function, one finds

$$W(x, x') = W_0(x, x') - \frac{1}{4\pi^2 a^2} \sum_{l=0}^{\infty} \frac{(2l+1) P_l(\cos\gamma)}{\sqrt{\sinh r \sinh r'}} \, e^{i(l+1/2)\pi} \int_{x_M}^{\infty} dx \, x \frac{\Gamma(x+l+1)}{\Gamma(x-l)}$$

$$\times \frac{Q_{x-1/2}^{-l-1/2}(u_0)}{P_{x-1/2}^{-l-1/2}(u_0)} P_{x-1/2}^{-l-1/2}(\cosh r) P_{x-1/2}^{-l-1/2}(\cosh r') \frac{\cosh\left(\sqrt{x^2 - x_M^2}\,\Delta t/a\right)}{\sqrt{x^2 - x_M^2}}, \quad (54)$$

where we have defined

$$x_M = \sqrt{M^2 a^2 + 1 - 6\xi}. \tag{55}$$

In formula (54), the first term on the right-hand side is given by

$$W_0(x, x') = \frac{1}{8\pi^2 a^3} \sum_{l=0}^{\infty} \frac{(2l+1) P_l(\cos\gamma)}{\sqrt{\sinh r \sinh r'}} \int_0^{\infty} dx \, x \sinh(\pi x)$$

$$\times |\Gamma(ix+l+1)|^2 P_{ix-1/2}^{-l-1/2}(\cosh r) P_{ix-1/2}^{-l-1/2}(\cosh r') \frac{e^{-i\omega(x)\Delta t}}{\omega(x)}. \tag{56}$$

This function does not depend on the sphere radius and is the Wightman function for a scalar field in the background spacetime described by the line element (35) when boundaries are absent. This can also be seen by the direct evaluation. Indeed, when boundaries are absent the eigenfunctions are still given by formula (42), where now the spectrum for z is continuous. In this case the corresponding part on the right of the orthonormalization condition (46) should be understood as the Dirac delta function. In the case $z = z'$ the normalization integral diverges and, hence, the main contribution comes from large values r. By using the asymptotic formulae for the associated Legendre functions for large values of the argument, we can see that

$$\int_1^{\infty} du \, P_{iz-1/2}^{-l-1/2}(u) P_{iz'-1/2}^{-l-1/2}(u) = \left| \frac{\Gamma(iz)}{\Gamma(l+1+iz)} \right|^2 \delta(z-z'). \tag{57}$$

By using this result for the normalization coefficient in the case when boundaries are absent one finds

$$C_\alpha = \frac{1}{\sqrt{2\omega a^3}} \left| \frac{\Gamma(l+1+iz)}{\Gamma(iz)} \right|, \tag{58}$$

and the eigenfunctions have the form (see also, [4, 26])

$$\varphi_\alpha(x) = \left| \frac{\Gamma(l+1+iz)}{\Gamma(iz)} \right| \frac{P_{iz-1/2}^{-l-1/2}(\cosh r)}{\sqrt{2\omega a^3 \sinh r}} Y_{lm}(\theta, \phi) \, e^{-i\omega t}. \tag{59}$$

Substituting these eigenfunctions into the mode-sum (38), for the corresponding Wightman function we find the formula which coincides with (56).

The case of a spherical boundary in the Minkowski spacetime is obtained in the limit $a \to \infty$, with fixed $ar = R$. In this limit one has $x_M = aM$. Introducing a new integration variable $y = x/a$, using formulae (24) and the asymptotic formula for the gamma function for large values of the argument, we find

$$W^{(M)}(x, x') = W_0^{(M)}(x, x') - \sum_{l=0}^{\infty} \frac{(2l+1) P_l(\cos\gamma)}{4\pi^2 \sqrt{RR'}} \int_M^{\infty} dy \, y$$

$$\times I_{l+1/2}(Ry) I_{l+1/2}(R'y) \frac{K_{l+1/2}(R_0 y)}{I_{l+1/2}(R_0 y)} \frac{\cosh(\sqrt{y^2 - M^2}\,\Delta t)}{\sqrt{y^2 - M^2}}. \tag{60}$$

This formula gives the positive frequency Wightman function inside a spherical shell with radius R_0 in the Minkowski bulk and is a special case of the general formula given in the first paper of [14] for a scalar field with Robin boundary conditions in an arbitrary number of spatial dimensions.

Having the Wightman function (54), we can evaluate the vacuum expectation value of the field squared taking the coincidence limit of the argument. Of course, this limit is divergent and some renormalization procedure is necessary. Here the important point is that for points outside the spherical shell the local geometry is the same as for the case without boundaries and, hence, the structure of the divergences is the same as well. This is also directly seen from formula (54), where the second term on the right-hand side is finite in the coincidence limit. Since in formula (54) we have already explicitly subtracted the boundary-free part, the renormalization is reduced to that for the geometry without boundaries. In this way for the renormalized vacuum expectation value of the field squared one has

$$
\langle \varphi^2 \rangle_{\text{ren}} = \langle \varphi^2 \rangle_{0,\text{ren}} - \sum_{l=0}^{\infty} \frac{e^{i(l+1/2)\pi}}{4\pi^2 a^2} \frac{(2l+1)}{\sinh r} \int_{x_M}^{\infty} dx \, x
$$
$$
\times \frac{\Gamma(x+l+1)}{\Gamma(x-l)} \frac{Q_{x-1/2}^{-l-1/2}(\cosh r_0)}{P_{x-1/2}^{-l-1/2}(\cosh r_0)} \frac{\left[P_{x-1/2}^{-l-1/2}(\cosh r)\right]^2}{\sqrt{x^2 - x_M^2}}, \tag{61}
$$

where the first term on the right-hand side is the corresponding quantity in the constant negative curvature space without boundaries and the second one is induced by the presence of the spherical shell. For large values x, the integrand in (61) behaves as $e^{-(r_0-r)x}/(2x \sinh r)$ and the integral is exponentially convergent at the upper limit for strictly interior points.

For $r \to 0$ one has $P_{x-1/2}^{-l-1/2}(\cosh r) \approx (r/2)^{l+1/2}/\Gamma(l+3/2)$, and in the boundary-induced part at the sphere center the $l = 0$ term contributes only

$$
\langle \varphi^2 \rangle_{\text{ren}} = \langle \varphi^2 \rangle_{0,\text{ren}} - \frac{1}{2\pi^2 a^2} \int_{x_M}^{\infty} dx \frac{x^2 (x^2 - x_M^2)^{-1/2}}{e^{2xr_0} - 1}, \qquad r = 0, \tag{62}
$$

where we have used formulae (19). Note that for a conformally coupled field the boundary-induced part in (62) coincides with the corresponding quantity for the sphere with radius $a r_0$ in the Minkowski bulk.

5. Conclusion

The associated Legendre functions are an important class of special functions that appear in a wide range of problems of mathematical physics. In the present paper, specifying the functions in the generalized Abel–Plana formula in the form (4), we have derived summation formula (15) for the series over the zeros of the associated Legendre function $P_{iz-1/2}^{\mu}(u)$ with respect to the degree. This formula is valid for functions $h(z)$ meromorphic in the right half-plane and obeying condition (9). Using formula (15), the difference between the sum over the zeros of the associated Legendre function and the corresponding integral is presented in terms of an integral involving the associated Legendre functions with real values of the degree plus residue terms. For a large class of functions $h(z)$ this integral converges exponentially fast and, in particular, is useful for numerical calculations. Frequently used two standard forms of the Abel–Plana formula are obtained as special cases of formula (15) with $\mu = -1/2$ and $\mu = 1/2$ and for an analytic function $h(z)$. Applying the summation formula for the series over the zeros of the function $P_{iz-1/2}^{\mu}(u \cosh(\eta/s))$ and taking the limit $s \to \infty$ we have obtained formula (26) for the summation of the series over the zeros of the Bessel function.

The latter is a special case of the formula, previously derived in [6]. Further, we specify the summation formula for two special cases of the order $\mu = -l$ and $\mu = -l - 1/2$ with l being a non-negative integer and give examples of the application of this formula. The associated Legendre functions with these values of the order arise as solutions of the waveequation on the background of constant curvature spaces in cylindrical and spherical coordinates.

In section 4 we consider a physical application of the summation formula. Namely, for a quantum scalar field we evaluate the positive frequency Wightman function and the vacuum expectation value of the field squared inside a spherical shell in a constant negative curvature space assuming that the field obeys the Dirichlet boundary condition on the shell. In spherical coordinates the radial part of the corresponding eigenfunctions contains the function $P_{iz-1/2}^{-l-1/2}(\cosh r)$ and the eigenfrequencies are expressed in terms of the zeros z_k by relation (45). As a result, the mode-sum for the Wightman function includes the summation over these zeros. For the evaluation of the corresponding series we apply summation formula (15) with the function $h(z)$ given by (53). The term with the first integral on the right-hand side of formula (15) corresponds to the Wightman function for the constant curvature space without boundaries and the term with the second integral is induced by the spherical boundary. For points away from the shell the latter is finite in the coincidence limit and can be directly used for the evaluation of the boundary induced part in the vacuum expectation value of the field squared. The latter is given by the second term on the right-hand side of formula (61). The renormalization is necessary for the boundary-free part only and this procedure is the same as that in quantum field theory without boundaries.

On the physical example considered we have demonstrated the advantages for the application of the Abel–Plana-type formulae in the evaluation of the expectation values of local physical observables in the presence of boundaries. For the summation of the corresponding mode-sums the explicit form of the eigenmodes is not necessary and the part corresponding to the boundary-free space is explicitly extracted. Further, the boundary-induced part is presented in the form of an integral which rapidly converges and is finite in the coincidence limit for points away from the boundary. In this way the renormalization procedure for local physical observables is reduced to that in quantum field theory without boundaries. Note that methods for the evaluation of global characteristics of the vacuum, such as total Casimir energy, in problems where the eigenmodes are given implicitly as zeros of a given function, are described in [27].

Acknowledgments

Appendix A. On zeros of the function $P_{iz-1/2}^{\mu}(u)$

In this appendix we show that the zeros $z = z_k$ are simple and real. By making use of the differential equation for the associated Legendre functions it can be seen that the following integration formula takes place:

$$\int du\, P_{\nu'}^{\mu}(u) P_{\nu}^{\mu}(u) = (1 - u^2) \frac{P_{\nu'}^{\mu}(u)\partial_u P_{\nu}^{\mu}(u) - P_{\nu}^{\mu}(u)\partial_u P_{\nu'}^{\mu}(u)}{\nu'(\nu'+1) - \nu(\nu+1)} + \text{const.} \tag{A.1}$$

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Taking the limit $\nu' \to \nu$ and applying the l'Hôpital's rule for the right-hand side, from this formula we find

$$\int \mathrm{d}u \left[P_\nu^\mu(u) \right]^2 = (1 - u^2) \frac{\left[\partial_\nu P_\nu^\mu(u) \right] \partial_u P_\nu^\mu(u) - P_\nu^\mu(u) \partial_\nu \partial_u P_\nu^\mu(u)}{2\nu + 1} + \text{const.} \qquad (A.2)$$

By taking into account the relation $P_{-\mathrm{i}z-1/2}^\mu(u) = P_{\mathrm{i}z-1/2}^\mu(u)$, we see that for real z one has $\left[P_{\mathrm{i}z-1/2}^\mu(u) \right]^2 = \left| P_{\mathrm{i}z-1/2}^\mu(u) \right|^2$. Hence, from formula (A.2) we find

$$\int_1^u \mathrm{d}v \left| P_{\mathrm{i}z-1/2}^\mu(v) \right|^2 = \frac{u^2 - 1}{2z} \left\{ \left[\partial_z P_{\mathrm{i}z-1/2}^\mu(u) \right] \partial_u P_{\mathrm{i}z-1/2}^\mu(u) - P_{\mathrm{i}z-1/2}^\mu(u) \partial_z \partial_u P_{\mathrm{i}z-1/2}^\mu(u) \right\}. \tag{A.3}$$

Here we have taken into account that for $u \to 1$ one has $P_{\mathrm{i}z-1/2}^\mu(u) \sim (u - 1)^{-\mu}$ and, hence, $\lim_{u \to 1} P_{\mathrm{i}z-1/2}^\mu(u) = 0$ for $\mu < 0$. From formula (A.3) it follows that $\left[\partial_z P_{\mathrm{i}z-1/2}^\mu(u) \right]_{z=z_k} \neq 0$, and, hence, the zeros z_k are simple.

Now let us show that under the conditions $u > 1$ and $\mu \leqslant 0$ all zeros of the function $P_{\mathrm{i}z-1/2}^\mu(u)$ are real. Suppose that $z = \lambda$ is a zero of $P_{\mathrm{i}z-1/2}^\mu(u)$ which is not real. As the function $P_{z-1/2}^\mu(u)$ has no real zeros (see, for instance, [28]), λ is not purely imaginary. If λ^* is the complex conjugate to λ, then it is also a zero of $P_{\mathrm{i}z-1/2}^\mu(u)$, because $P_{\mathrm{i}\lambda^*-1/2}^\mu(v) = \left[P_{\mathrm{i}\lambda-1/2}^\mu(v) \right]^*$. As a result, from formula (A.1) we find

$$\int_1^u \mathrm{d}v \, P_{\mathrm{i}\lambda^*-1/2}^\mu(v) P_{\mathrm{i}\lambda-1/2}^\mu(v) = 0. \tag{A.4}$$

We have obtained a contradiction, since the integrand on the left-hand side is positive. Hence the number λ cannot exist and the function $P_{\mathrm{i}z-1/2}^\mu(u)$ has no zeros which are not real.

From the asymptotic formula (B.4) for the function $P_{\mathrm{i}z-1/2}^\mu(u)$ (see appendix B) we obtain the asymptotic expression for large zeros:

$$z_k \sim (\pi k - \pi \mu/2 - \pi/4)/\eta. \tag{A.5}$$

Note that this result can also be obtained by taking into account that for large values z from (23) one has $P_{\mathrm{i}z-1/2}^{-\mu}(\cosh(\eta)) \approx z^{-\mu} J_\mu(\eta z)$ and using the asymptotic form for the zeros of the Bessel function (see, for instance, [22]).

Appendix B. Asymptotics of the associated Legendre functions

In this appendix we consider asymptotic expressions for the associated Legendre functions for large values of the degree. As a starting point we use the formula

$$Q_{z-1/2}^\mu(\cosh \eta) = \sqrt{\pi} \, \mathrm{e}^{\mathrm{i}\mu\pi} \frac{\Gamma(1/2 + z + \mu)}{\Gamma(1 + z)} \frac{(1 - \mathrm{e}^{-2\eta})^\mu}{\mathrm{e}^{(z+1/2)\eta}} F(1/2 + \mu, 1/2 + z + \mu; 1 + z; \mathrm{e}^{-2\eta}). \tag{B.1}$$

Using the linear transformation formula 15.3.4 from [22] for the hypergeometric function, the expression for the function $Q_{z-1/2}^\mu(\cosh \eta)$ is presented in the form

$$Q_{z-1/2}^\mu(\cosh \eta) = \sqrt{\pi} \mathrm{e}^{\mathrm{i}\mu\pi} \frac{\Gamma(1/2 + z + \mu)}{\Gamma(1 + z)} \frac{\mathrm{e}^{-z\eta}}{\sqrt{2 \sinh \eta}} F(1/2 + \mu, 1/2 - \mu; 1 + z; 1/(1 - \mathrm{e}^{2\eta})). \tag{B.2}$$

Now, by using the result that for large $|c|$ one has $F(a, b; c; z) = 1 + O(1/|c|)$, from (B.2) the asymptotic formula for the function $Q_{z-1/2}^\mu(\cosh \eta)$ is obtained for large values $|z|$. The

corresponding formula for the function $P^{\mu}_{z-1/2}(\cosh\eta)$ is obtained by using the relation

$$\pi e^{i\mu\pi} \sin(\pi z) P^{\mu}_{z-1/2}(\cosh\eta)$$
$$= \cos[\pi(z-\mu)]Q^{\mu}_{-z-1/2}(\cosh\eta) - \cos[\pi(z+\mu)]Q^{\mu}_{z-1/2}(\cosh\eta). \tag{B.3}$$

In this way we obtain the following formulae,

$$P^{\mu}_{z-1/2}(\cosh\eta) \sim \sqrt{\frac{2}{\pi}} \frac{y^{\mu-1/2}}{\sqrt{\sinh\eta}} \sin(\eta y - i\eta x + \pi\mu/2 + \pi/4),$$

$$Q^{\mu}_{z-1/2}(\cosh\eta) \sim e^{i\mu\pi} \sqrt{\frac{\pi}{2}} \frac{y^{\mu-1/2}}{\sqrt{\sinh\eta}} \exp[-\eta x - i(\eta y - \pi\mu/2 + \pi/4)], \tag{B.4}$$

in the limit $y \to +\infty$, $z = x + iy$, and the formulae

$$P^{\mu}_{z-1/2}(\cosh\eta) \sim \frac{x^{\mu-1/2}}{\sqrt{2\pi\sinh\eta}} e^{\eta x + i\eta y},$$

$$Q^{\mu}_{z-1/2}(\cosh\eta) \sim \sqrt{\frac{\pi}{2}} e^{i\mu\pi} \frac{x^{\mu-1/2}}{\sqrt{\sinh\eta}} e^{-\eta x - i\eta y}, \tag{B.5}$$

in the limit $x \to +\infty$.

Now let us consider the asymptotics of the functions $P^{-\mu}_{i\nu-1/2}(\cosh(\eta/\nu))$ and $Q^{-\mu}_{\pm i\nu-1/2}(\cosh(\eta/\nu))$ as $\nu \to +\infty$. These asymptotics are obtained in the way similar to that used in [25] for formulae (24). Our starting point is the formula

$$P^{-\mu}_{i\nu-1/2}(\cosh(\eta/\nu)) = \frac{\tanh^{\mu}(\eta/2\nu)}{\Gamma(1+\mu)} F(1/2 - i\nu, 1/2 + i\nu; 1 + \mu; -\sinh^2(\eta/2\nu)), \tag{B.6}$$

relating the associated Legendre function to the hypergeometric function. From the definition of the hypergeometric function it is not difficult to see that

$$\lim_{\nu\to+\infty} F(1/2 - i\nu, 1/2 + i\nu; 1 + \mu; -\sinh^2(\eta/2\nu)) = \Gamma(1+\mu)(2/\eta)^{\mu} J_{\mu}(\eta). \tag{B.7}$$

Combining (B.6) and (B.7) we obtain formula (23). The corresponding formula for the functions $Q^{-\mu}_{\pm i\nu-1/2}(\cosh(\eta/\nu))$ is obtained by making use of the relation

$$\frac{2}{\pi} \sin(\mu\pi) e^{i\mu\pi} Q^{-\mu}_{\pm i\nu-1/2}(u) = \frac{\Gamma(\pm i\nu - \mu + 1/2)}{\Gamma(\pm i\nu + \mu + 1/2)} P^{\mu}_{i\nu-1/2}(u) - P^{-\mu}_{i\nu-1/2}(u), \tag{B.8}$$

and formula (23). In this way we find

$$\lim_{\nu\to+\infty} \nu^{\mu} e^{i\mu\pi} Q^{-\mu}_{\pm i\nu-1/2}(\cosh(\eta/\nu)) = \pi \frac{e^{\mp i\mu\pi} J_{-\mu}(\eta) - J_{\mu}(\eta)}{2\sin(\mu\pi)}, \tag{B.9}$$

or in the equivalent form

$$\lim_{\nu\to+\infty} \nu^{\mu} e^{i\mu\pi} Q^{-\mu}_{i\nu-1/2}(\cosh(\eta/\nu)) = -\frac{\pi i}{2} e^{-i\mu\pi} H^{(2)}_{\mu}(\eta),$$

$$\lim_{\nu\to+\infty} \nu^{\mu} e^{i\mu\pi} Q^{-\mu}_{-i\nu-1/2}(\cosh(\eta/\nu)) = \frac{\pi i}{2} e^{i\mu\pi} H^{(1)}_{\mu}(\eta), \tag{B.10}$$

where $H^{(1,2)}_{\mu}(\eta)$ are the Hankel functions.

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